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Extremal Vehicle Reorientation Maneuvers: Symmetries and Group Properties

Spiro Bocvarov,* Frederick H. Lutze,† and Eugene M. Cliff‡
Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

The setting for this study is time-optimal reorientation maneuvers for a rigid-body. Analysis of the dynamic model reveals certain symmetries among solutions for a class of rest-to-rest maneuvers. An operator-theoretic approach is used to formalize the analysis and the set of operators is shown to form a commutative group. For extremal trajectories, a second group of transformations is identified: these are mappings between the adjoint variables. The two groups are shown to be homomorphic. A particular subgroup of the adjoints maps is highlighted and used to generate several distinct extremal trajectories from a single known one. A number of extremal trajectories, shown graphically, supplement and illustrate the analytical considerations.

I. Introduction

THIS study occurred at the initial stages of a project focused on understanding minimum-time fuselage reorientation maneuvers for combat aircraft.[1,2] A mathematical model of a combat aircraft was developed that included the aerodynamic and thrust-vectoring (propulsive) moment controls. By applying necessary conditions for optimality,[3-5] the optimal control problem was cast into the form of a numerical multipoint boundary-value problem. Since these are extremely difficult to solve, a step-by-step approach was adopted. First, a few extremal solutions were obtained for an aircraft in vacuum, with propulsive control moments only (the reader may think of a spacecraft, as there has been a great amount of work done in the area of optimal spacecraft reorientation maneuvering[6-9]). These solutions were used as starting points for homotopy procedures wherein the effect of the aerodynamic moments was added progressively and in stages. In such homotopy studies a particular regular extremal family may disappear. The boundary-value problem may cease to have a solution, or an extremal family could become singular. Hence it was important to find a lot of extremal solutions (hopefully, all of them) for the problem of the vehicle in vacuum. The main point of this paper is that, for our system, it was possible to derive new extremals from known ones, without additional numerical calculations.

The mathematical model, shown in Sec. II, corresponds to the situation just described of the vehicle in vacuum. Some structural characteristics of the mathematical model are pointed out, and the 90-deg roll maneuver (which the discussion in this paper is related to) is defined. In Sec. III a set of symmetric trajectories is defined, and appropriate relations derived. The term "symmetric" does not refer to a particular geometric symmetry, but to a more general concept defined algebraically. An operator-transformation description is adopted for convenience, and the concept of "composition of symmetries" is defined. In Sec. IV we show that for symmetric extremal trajectories the corresponding adjoint variables are correlated. Finally, in Sec. V a group[10,11] of adjoint variables transformations is established. The analytical relations are applied to a few extremal solutions, and the derived trajectories shown graphically.

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*Research Associate, Aerospace and Ocean Engineering Department. Member AIAA.

†Professor, Aerospace and Ocean Engineering Department. Associate Fellow AIAA.

‡Reynolds Metals Professor, Aerospace and Ocean Engineering Department. Associate Fellow AIAA.

II. Mathematical Modeling

A. Attitude Model

An appropriately scaled mathematical model for attitude motions of a vehicle[1,2,12] is given by Eqs. (1). The state vector is $\bar{x} = (\alpha, \beta, \mu, p, q, r)^*$ (a superscript asterisk denotes transpose). The constant parameters are given by $a_i = M_i/I_i$ $(i,j,k = x,y,z)$ and, for example $J_x = (I_y - I_z)/I_x$. These parameters are related to the vehicle's inertial properties and propulsive power. M_i stands for the maximal propulsive-moment control power with respect to the i axis $(i = x,y,z)$ and δ_p, δ_q, and δ_r represent the normalized controls. The angles α and β are the customary aerodynamic angles for aircraft. One can imagine μ as a wind-axes roll angle, the wind-axis reference frame $x^w y^w z^w$ being constrained to roll about the x^h axis only, and assumed to coincide with the local-horizontal reference frame when $\mu = 0$. Then β is the angle between the x^h axis and its projection on the body-axes $x^b z^b$ plane, and α is the angle between that projection and the x^b axis[12] (one can imagine the vehicle velocity vector, which is not of interest in our considerations, coinciding with the x^h axis).

$$
\begin{aligned}
\dot{\alpha} &= q - (p \cos\alpha + r \sin\alpha)\tan\beta \\
\dot{\beta} &= p \sin\alpha - r \cos\alpha \\
\dot{\mu} &= (p \cos\alpha + r \sin\alpha)/\cos\beta \\
\dot{p} &= J_x q r + a_x \delta_p \\
\dot{q} &= J_y r p + a_y \delta_q \\
\dot{r} &= J_z p q + a_z \delta_r
\end{aligned} \tag{1}
$$

It should be noted here that the dynamics of the attitude angles (α, β, μ) depend on the angular rates $\bar{\omega} = (p, q, r)^*$ in the body-fixed reference frame $x^b y^b z^b$, and the attitude angles themselves. The angular rates appear linearly in these kinematic equations. The Euler rigid-body dynamic equations are nonlinear, due to the gyroscopic terms. However, the control variables $(\delta_p, \delta_q, \delta_r)$ appear in the dynamic equations in a linear manner. The system is autonomous, and the variable μ is ignorable.

B. Attitude Maneuvers

One particular rest-to-rest reorientation maneuver $[\bar{\omega}(0) = \bar{\omega}(t_f) = 0]$, to which the discussion in the remaining sections

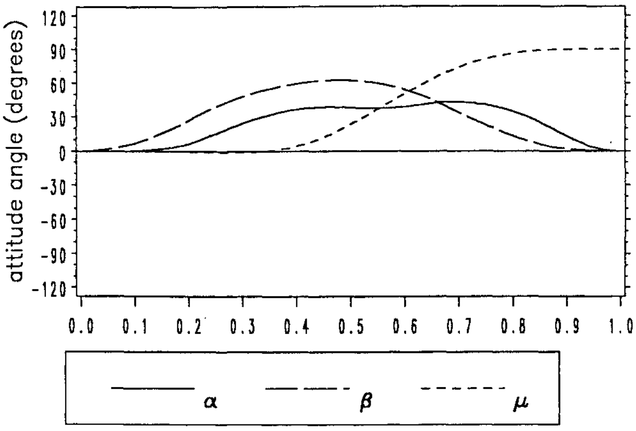

Fig. 1a Family A: trajectory M_0 attitude angles.

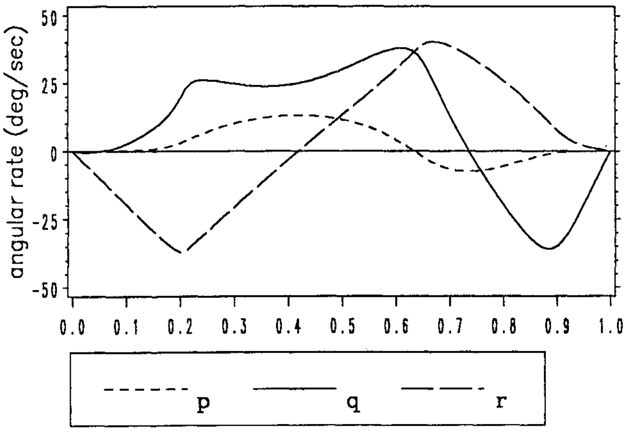

Fig. 1b Family A: trajectory M_0 angular rates.

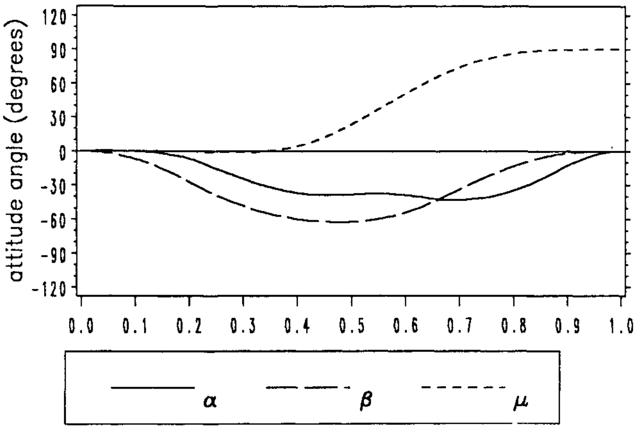

Fig. 2a Family A: trajectory M_a attitude angles.

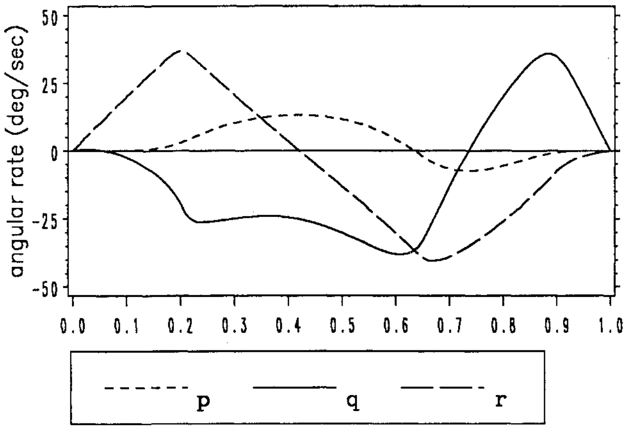

Fig. 2b Family A: trajectory M_a angular rates.

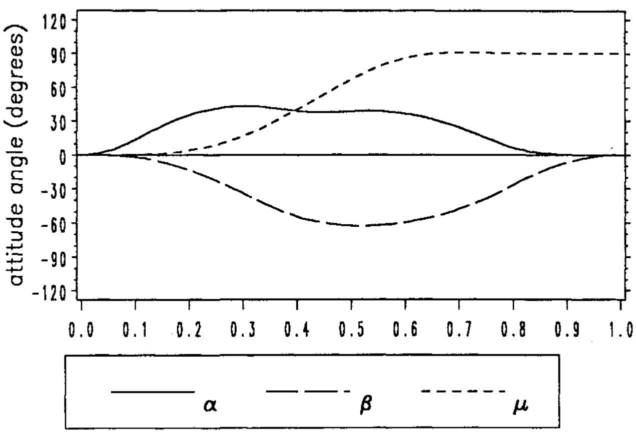

Fig. 3a Family A: trajectory M_{pt} attitude angles.

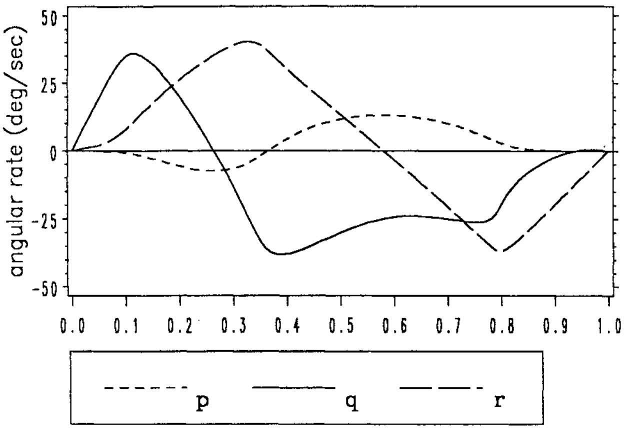

Fig. 3b Family A: trajectory M_{pt} angular rates.

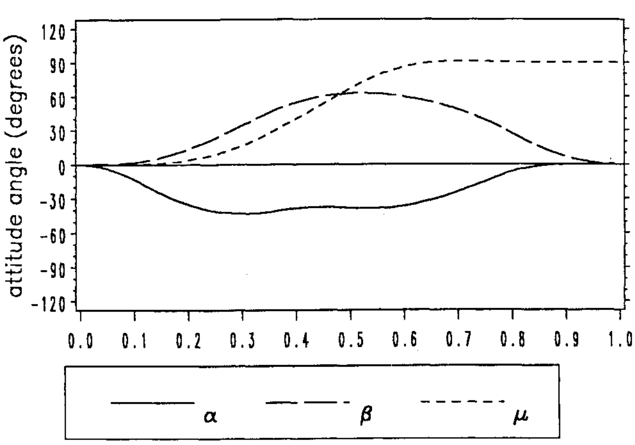

Fig. 4a Family A: trajectory M_{apt} attitude angles.

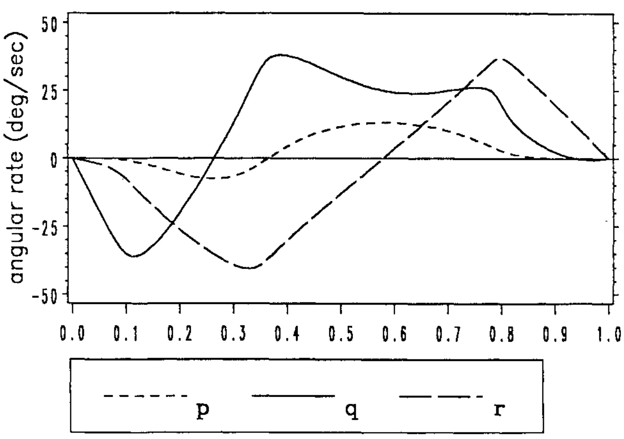

Fig. 4b Family A: trajectory M_{apt} angular rates.

will be related, is the +90-deg roll. By performing this maneuver, the vehicle effectively rolls around the x^b axis for 90 deg, in a positive sense. More precisely, for a +90-deg roll maneuver the initial and final state can be given by $\bar{x}_0 = (0\text{ deg}, 0\text{ deg}, \mu_0, 0, 0, 0)^*$ and $\bar{x}_f = (0\text{ deg}, 0\text{ deg}, \mu_f, 0, 0, 0)^*$ with $\mu_f - \mu_0 \equiv \Delta\mu = +90$ deg. Note that because μ is an ignorable variable, the same control $\bar{u}(\cdot) = (\delta_p, \delta_q, \delta_r)^*(\cdot)$ that rolls the vehicle from $(\alpha,\beta,\mu)^*(0) = (0\text{ deg}, 0\text{ deg}, 0\text{ deg})^*$ to $(\alpha,\beta,\mu)^*(T) = (0\text{ deg}, 0\text{ deg}, 90\text{ deg})^*$ (T denotes maneuvering time) will roll the vehicle for $\Delta\mu = +90$ deg, no matter what the initial value of μ is. For simplicity we assume that $\mu_0 = 0$, so that at time zero the body-fixed and the local-horizontal reference frames coincide.

As stated in the Introduction, the model considered corresponds to an aircraft with propulsive control moments. The thrust-vectoring system cannot generate large roll moments (the extremal trajectories shown in Figs. 1–8 correspond to this situation, for $a_x = 0$ [Eqs. (1)]. When $a_x = 0$, the evolution of the roll-rate p can be influenced through the gyroscopic term ($J_x qr$) only. For the case $a_x \neq 0$ there is a trivial extremal solution—pure roll—although this maneuver might not be the optimal one.[13,14] In the rest of the discussion we make no assumptions about a_x.

III. Symmetries and Compositions

The considerations described in this section were inspired by analyses of geometric symmetries of vehicle attitude maneuvers. The concept of symmetry was further extended and generalized algebraically. Thus, the term "symmetry" of two trajectories is used in this paper to indicate that some of the state and control variables of the trajectories are related by identities, sign change, and/or time reversal, as defined further. We shall define three particular symmetries.

A. Axial Symmetry

Let us assume that a +90-deg roll maneuver

$$\bar{x}(0) = (0\text{ deg}, 0\text{ deg}, 0\text{ deg}, 0, 0, 0)^*$$
$$\bar{x}(T) = (0\text{ deg}, 0\text{ deg}, 90\text{ deg}, 0, 0, 0)^* \tag{2}$$

is performed by applying controls $\bar{u}^0(t)$ and that the corresponding trajectory is given by $\bar{x}^0(t)$

$$\bar{x}^0(t) = (\alpha,\beta,\mu,p,q,r)^*(t)$$
$$\bar{u}^0(t) = (\delta_p,\delta_q,\delta_r)^*(t) \tag{3}$$

for $0 \leq t \leq T$. We will denote this trajectory by M_0, and refer to it as the nominal one. The question of interest is: does another trajectory exist, symmetric to M_0 in the general algebraic sense, for the same maneuver? Before pursuing this question it is helpful to note that we use the term "maneuver" to denote initial and final states ("endpoints"). The term "trajectory" will denote the path in state-space that connects the given endpoint conditions, along with the associated (maneuvering) time. A given maneuver can be performed by many trajectories, and here we seek some simply related trajectories.

Each of the 32 possible combinations

$$\alpha^s(t) = \pm\alpha(t), \qquad \beta^s(t) = \pm\beta(t), \qquad \mu^s(t) = +\mu(t)$$
$$p^s(t) = \pm p(t), \qquad q^s(t) = \pm q(t), \qquad r^s(t) = \pm r(t) \tag{4}$$

satisfies the initial and final conditions in Eq. (2). Note that a characteristic of this symmetry is the requirement $\mu^s(t) = +\mu(t)$.

Effectively, the question is whether for any of the state combinations in Eq. (4), other than the original one, there is a control vector $\bar{\delta}^s(t) = (\delta_p^s, \delta_q^s, \delta_r^s)^*(t)$ among

$$\delta_p^s(t) = \pm\delta_p(t)$$
$$\delta_q^s(t) = \pm\delta_q(t) \tag{5}$$
$$\delta_r^s(t) = \pm\delta_r(t)$$

such that Eqs. (1) are satisfied. As a trivial exercise, it can be found that there is only one such trajectory. We will denote it M_a, and refer to it as the axially symmetric trajectory to the nominal one (M_0). It is given by

$$\bar{x}^a(t) = (-\alpha, -\beta, \mu, p, -q, -r)^*(t)$$
$$\bar{u}^a(t) = (\delta_p, -\delta_q, -\delta_r)^*(t) \tag{6}$$

As a remark, it is the nonlinearity of the dynamic equations (the gyroscopic terms) that allow for only one symmetry of the type required [$\mu^a(t) = +\mu(t)$]. Also, the term "axial symmetry" is used for convenience: one can easily visualize that in the nominal and the axially symmetric trajectory only the vehicle's x^b axis is geometrically symmetric with respect to the x^h axis.

B. Plane Symmetry

Analogously to the axial symmetry, we would like to know if there exists a symmetric trajectory with the characteristic $\mu^s(t) = -\mu(t)$. The answer is positive. Not surprisingly, there are two such trajectories. The first trajectory, denoted M_p, is given by

$$\bar{x}^p(t) = (\alpha, -\beta, -\mu, -p, q, -r)^*(t)$$
$$\bar{u}^p(t) = (-\delta_p, \delta_q, -\delta_r)^*(t) \tag{7}$$

This trajectory is geometrically symmetric to the nominal one, with respect to the $x^h z^h$ plane. The other trajectory, denoted by M_{ap}, is specified by

$$\bar{x}^{ap}(t) = (-\alpha, \beta, -\mu, -p, -q, r)^*(t)$$
$$\bar{u}^{ap}(t) = (-\delta_p, -\delta_q, \delta_r)^*(t) \tag{8}$$

One can easily verify that M_{ap} is actually axially symmetric to M_p (as defined in Sec. III.A.).

C. Time Symmetry

The following problem provides motivation for examining time symmetries: once a maneuver has been performed, can we return the vehicle to its original position by following the same trajectory, but in the reversed order [$\bar{x}^r(t) = \bar{x}^0(T-t)$]? By examining the state dynamics we can see that the answer is negative. The obstacle for such reversed trajectories to exist, by simply applying control reversed in time [$\bar{u}^r(t) = -\bar{u}^0(T-t)$], are the nonlinearities in the dynamic equations (the gyroscopic terms). However, if we allow for generalized algebraic symmetries (sign changes in the states and the controls), then the following trajectory will accomplish the task:

$$\bar{x}^t(t) = (\alpha, \beta, \mu - 90\text{ deg}, -p, -q, -r)^*(T-t)$$
$$\bar{u}^t(t) = (-\delta_p, -\delta_q, -\delta_r)^*(T-t) \tag{9}$$

We will denote the preceding time-symmetric trajectory by M_t. Indeed, there is one more time-symmetric trajectory that will accomplish the desired task. It is specified by

$$\bar{x}^{at}(t) = (-\alpha, -\beta, \mu - 90\text{ deg}, -p, q, r)^*(T-t)$$
$$\bar{u}^{at}(t) = (-\delta_p, \delta_q, \delta_r)^*(T-t) \tag{10}$$

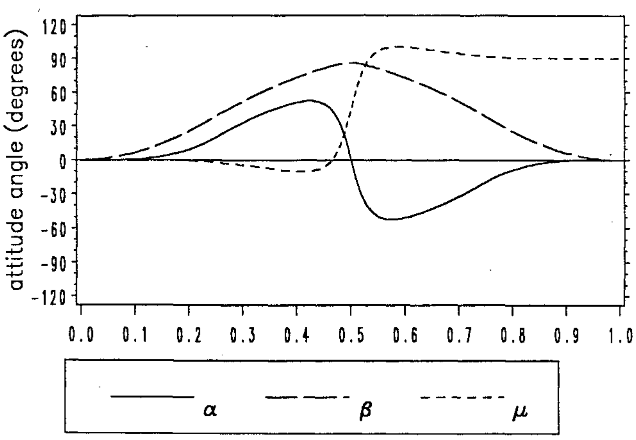

Fig. 5a Family B: trajectory $M_0(=M_{pt})$ attitude angles.

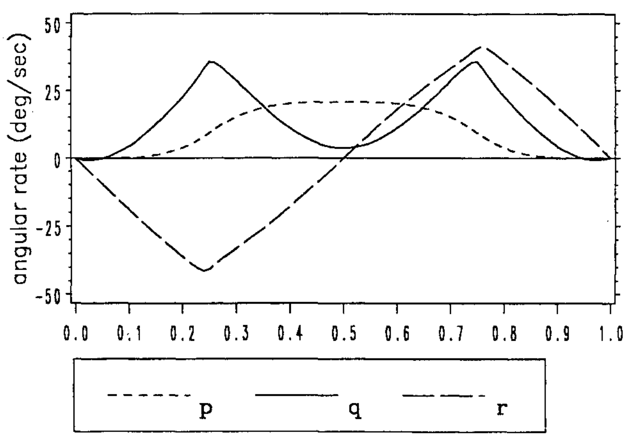

Fig. 5b Family B: trajectory $M_0(=M_{pt})$ angular rates.

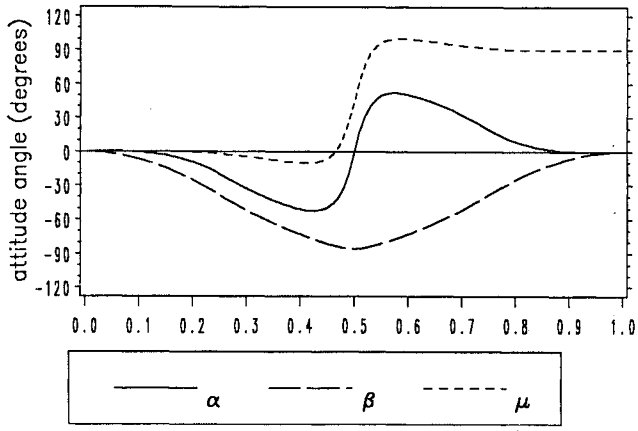

Fig. 6a Family B: trajectory $M_a(=M_{apt})$ attitude angles.

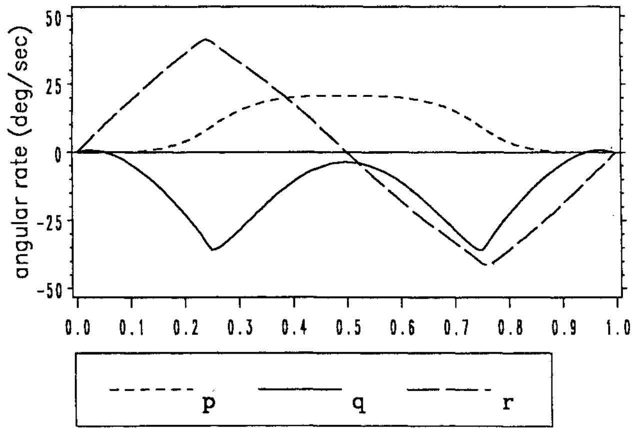

Fig. 6b Family B: trajectory $M_a(=M_{apt})$ angular rates.

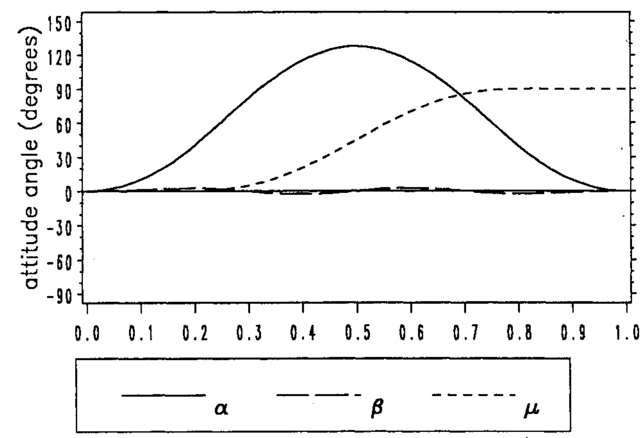

Fig. 7a Family C: trajectory $M_0(=M_{pt})$ attitude angles.

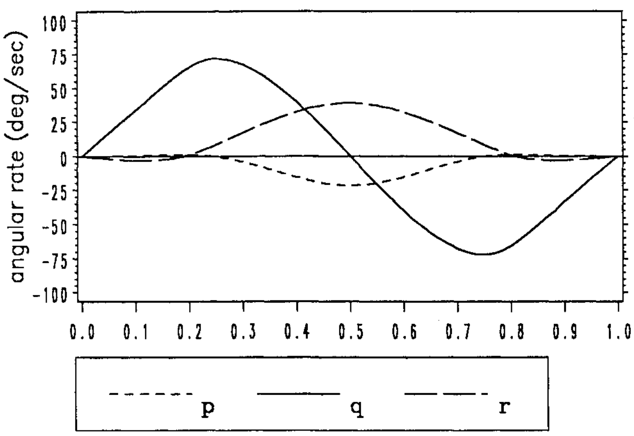

Fig. 7b Family C: trajectory $M_0(=M_{pt})$ angular rates.

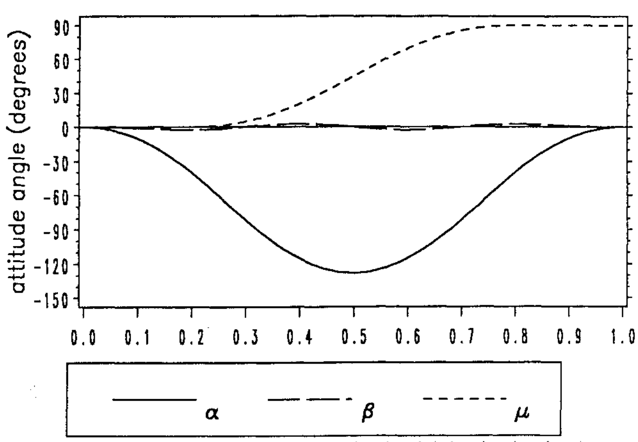

Fig. 8a Family C: trajectory $M_a(=M_{apt})$ attitude angles.

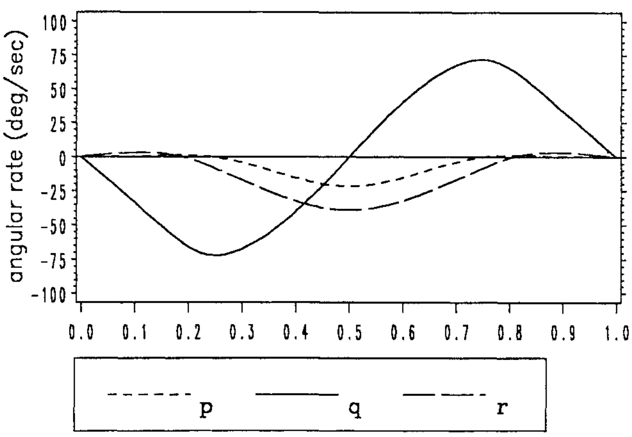

Fig. 8b Family C: trajectory $M_a(=M_{apt})$ angular rates.

This trajectory will be denoted M_{at}. It can be shown that M_{at} is axially symmetric to M_t.

D. Composition of Symmetries

One can think of the M_{ap} trajectory (8), which is axially symmetric to M_p, as being obtained from the nominal one (M_0) by a composition of symmetry operations. With each of the M_a, M_p, M_{ap}, M_t, and M_{tp} trajectories we can associate a linear operator, denoted $\mathfrak{M}_a$, $\mathfrak{M}_p$, $\mathfrak{M}_{ap}$, $\mathfrak{M}_t$, and $\mathfrak{M}_{tp}$, respectively, such that, for example,

$$\begin{bmatrix} \bar{x}^a(\cdot) \\ \bar{u}^a(\cdot) \end{bmatrix} = \mathfrak{M}_a \begin{bmatrix} \bar{x}^0(\cdot) \\ \bar{u}^0(\cdot) \end{bmatrix} \tag{11}$$

In this sense, $\mathfrak{M}_0 = \mathfrak{I}$ (the identity). It is a trivial exercise to show that $\mathfrak{M}_{ap} = \mathfrak{M}_a \mathfrak{M}_p = \mathfrak{M}_p \mathfrak{M}_a$. Similarly, $\mathfrak{M}_{at} = \mathfrak{M}_a \mathfrak{M}_t = \mathfrak{M}_t \mathfrak{M}_a$. Commutativity is preserved. Also, $\mathfrak{M}_a \mathfrak{M}_a = \mathfrak{M}_p \mathfrak{M}_p = \mathfrak{M}_t \mathfrak{M}_t = \mathfrak{M}_0$. It is in this sense that we can talk of composition of symmetry transformations. Here is another example.

Both M_t and M_{at} are -90-deg roll maneuvers. They return the vehicle from $\bar{x}(T) = (0 \text{ deg}, 0 \text{ deg}, +90 \text{ deg}, 0, 0, 0)^*$ to $\bar{x}(0) = (0 \text{ deg}, 0 \text{ deg}, 0 \text{ deg}, 0, 0, 0)^*$. However, since μ is an ignorable variable, both M_t and M_{at} can also reorient the vehicle from $\bar{x}(0) = (0 \text{ deg}, 0 \text{ deg}, 0 \text{ deg}, 0, 0, 0)^*$ to $\bar{x}(T) = (0 \text{ deg}, 0 \text{ deg}, -90 \text{ deg}, 0, 0, 0)^*$. This fact helps us generate other trajectories by composition. Intuitively, it is clear that if we apply plane symmetry to M_t we get a trajectory (denoted M_{pt}) that performs a $+90$-deg roll maneuver, as the nominal one (M_0) does

$$\begin{aligned} \bar{x}^{pt}(t) &= (+\alpha, \ -\beta, \ -\mu + 90 \text{ deg}, \ +p, \ -q, \ +r)^*(T-t) \\ \bar{u}^{pt}(t) &= (-\delta_p, \ +\delta_q, \ -\delta_r)^*(T-t) \end{aligned} \tag{12}$$

It can be shown that $\mathfrak{M}_{pt} = \mathfrak{M}_p \mathfrak{M}_t = \mathfrak{M}_t \mathfrak{M}_p$.

IV. Extremal Trajectories

A. Adjoint Equations

According to results from optimal-control theory,[3-5] the minimum-time motion for a given reorientation maneuver is an extremal trajectory. For the problem of interest, the variational Hamiltonian $H = -1 + \langle \bar{\lambda}, \dot{\bar{x}} \rangle$ is

$$\begin{aligned} H = {} & -1 + \lambda_\alpha \big[q - (p \cos\alpha + r \sin\alpha) \tan\beta \big] \\ & + \lambda_\beta \big[p \sin\alpha - r \cos\alpha \big] + \lambda_\mu \big[(p \cos\alpha + r \sin\alpha)/\cos\beta \big] \\ & + \lambda_p \left[a_x \delta_p + J_x qr \right] + \lambda_q \left[a_y \delta_q + J_y rp \right] + \lambda_r \left[a_z \delta_r + J_z pq \right] \end{aligned} \tag{13}$$

An extremal control $\bar{u}(\cdot) = (\delta_p, \delta_q, \delta_r)^*(\cdot)$ is such that the variational Hamiltonian is minimized for $0 \le t \le T$, where T is the extremal maneuvering time.

The dynamics of the vector of adjoint variables $\bar{\lambda} = (\lambda_\alpha, \lambda_\beta, \lambda_\mu, \lambda_p, \lambda_q, \lambda_r)^*$ is given by

$$\begin{aligned} \dot{\lambda}_\alpha &= -\lambda_\alpha (+p \sin\alpha - r \cos\alpha) \tan\beta \\ &\quad - \lambda_\beta (+p \cos\alpha + r \sin\alpha) - \lambda_\mu (-p \sin\alpha + r \cos\alpha)/\cos\beta \\ \dot{\lambda}_\beta &= +\lambda_\alpha (+p \cos\alpha + r \sin\alpha)/\cos\beta)^2 \\ &\quad - \lambda_\mu (+p \cos\alpha + r \sin\alpha) \sin\beta/(\cos\beta)^2 \\ \dot{\lambda}_\mu &= 0 \\ \dot{\lambda}_p &= -\lambda_q J_y r - \lambda_r J_z q + \lambda_\alpha \cos\alpha \tan\beta \\ &\quad - \lambda_\beta \sin\alpha - \lambda_\mu \cos\alpha/\cos\beta \\ \dot{\lambda}_q &= -\lambda_r J_z p - \lambda_p J_x r - \lambda_\alpha \\ \dot{\lambda}_r &= -\lambda_p J_x q - \lambda_q J_y p + \lambda_\alpha \sin\alpha \tan\beta \\ &\quad + \lambda_\beta \cos\alpha - \lambda_\mu \sin\alpha/\cos\beta \end{aligned} \tag{14}$$

Significant for our further considerations is the fact that the adjoint variables appear in a linear manner in the right-hand side of Eqs. (14). Also, the variational Hamiltonian is linear in both $\bar{\lambda}$ and $\dot{\bar{x}}$.

B. Transformation of the Adjoints

The discussion in Secs. III.A.–III.D. pertains to any trajectory of the dynamic system (1) that satisfies the initial and final state (2). In what follows we focus our interest on extremal trajectories only. Given that the nominal trajectory is an extremal, the question of interest is whether the identified symmetric trajectories are also extremal. In other words, does there exist an adjoint vector for each of the symmetric trajectories, such that the symmetric trajectory controls maximize the corresponding Hamiltonian? The answer is positive. There is a unique adjoint vector corresponding to each of the symmetric trajectories. The easiest way to find the desired adjoint variables is to check symmetric combinations of the original adjoint variables. This check is easy to perform due to the linear dependence of the Hamiltonian on $\bar{\lambda}$ and $\bar{u}$, and the homogeneity of the adjoint system. We note that the fact that the symmetric trajectories are extremal for our dynamic system (1) is not accidental. Interpreting the adjoint variables as sensitivity functions, one could deduce the desired adjoints from physical considerations.

These are the adjoint variables for each of the axial-, plane-, and time-symmetric trajectories, respectively:

$$\bar{\lambda}_a(t) = (-\lambda_\alpha, \ -\lambda_\beta, \ \lambda_\mu, \ \lambda_p, \ -\lambda_q, \ -\lambda_r)^*(t) \tag{15a}$$

$$\bar{\lambda}_p(t) = (\lambda_\alpha, \ -\lambda_\beta, \ -\lambda_\mu, \ -\lambda_p, \ \lambda_q, \ -\lambda_r)^*(t) \tag{15b}$$

$$\bar{\lambda}_t(t) = (-\lambda_\alpha, \ -\lambda_\beta, \ -\lambda_\mu, \ \lambda_p, \ \lambda_q, \ \lambda_r)^*(T-t) \tag{15c}$$

V. Symmetries and Group Properties

Next, we extend the ideas on composition of symmetry transformations, introduced in Sec. III.D. to (regular) extremal trajectories. A regular extremal trajectory is uniquely specified by the maneuvering time T (our dynamic system is autonomous, so that we assume all of our maneuvers start at time $t = 0$) and the initial value of the adjoint vector $\bar{\lambda}(0)$. Thus, instead of examining the operation composition of symmetry transformations, we can equivalently examine the operation composition of adjoint variables transformations. We can associate linear operators $\mathfrak{L}_a$, $\mathfrak{L}_p$, and $\mathfrak{L}_t$ with each of the transformed adjoint vectors (15a), (15b), and (15c), respectively. The operator corresponding to the nominal extremal trajectory adjoint vector is the identity map $\mathfrak{L}_0 \equiv \mathfrak{I}$. A systematic examination of all of the possible combinations of compositions reveals a special structure. We list here all of the essentially different adjoint vectors that can be obtained by compositions:

$$\begin{aligned} \bar{\lambda}_0(t) &= (+\lambda_\alpha, \ +\lambda_\beta, \ +\lambda_\mu, \ +\lambda_p, \ +\lambda_q, \ +\lambda_r)^*(t) \\ \bar{\lambda}_a(t) &= (-\lambda_\alpha, \ -\lambda_\beta, \ +\lambda_\mu, \ +\lambda_p, \ -\lambda_q, \ -\lambda_r)^*(t) \\ \bar{\lambda}_p(t) &= (+\lambda_\alpha, \ -\lambda_\beta, \ -\lambda_\mu, \ -\lambda_p, \ +\lambda_q, \ -\lambda_r)^*(t) \\ \bar{\lambda}_{ap}(t) &= (-\lambda_\alpha, \ +\lambda_\beta, \ -\lambda_\mu, \ -\lambda_p, \ -\lambda_q, \ +\lambda_r)^*(t) \\ \bar{\lambda}_t(t) &= (-\lambda_\alpha, \ -\lambda_\beta, \ -\lambda_\mu, \ +\lambda_p, \ +\lambda_q, \ +\lambda_r)^*(T-t) \\ \bar{\lambda}_{at}(t) &= (+\lambda_\alpha, \ +\lambda_\beta, \ -\lambda_\mu, \ +\lambda_p, \ -\lambda_q, \ -\lambda_r)^*(T-t) \\ \bar{\lambda}_{pt}(t) &= (-\lambda_\alpha, \ +\lambda_\beta, \ +\lambda_\mu, \ -\lambda_p, \ +\lambda_q, \ -\lambda_r)^*(T-t) \\ \bar{\lambda}_{apt}(t) &= (+\lambda_\alpha, \ -\lambda_\beta, \ +\lambda_\mu, \ -\lambda_p, \ -\lambda_q, \ +\lambda_r)^*(T-t) \end{aligned} \tag{16}$$

We can consolidate our results by noting that the set of elements $\mathfrak{L}_0$, $\mathfrak{L}_a$, $\mathfrak{L}_p$, $\mathfrak{L}_t$, $\mathfrak{L}_{ap}$, $\mathfrak{L}_{at}$, $\mathfrak{L}_{pt}$, and $\mathfrak{L}_{apt}$ with the operation composition (multiplication) is a commutative

Table 1 Subgroup G_L^+

G_L^+	$\mathcal{L}_0$	$\mathcal{L}_a$	$\mathcal{L}_{pt}$	$\mathcal{L}_{apt}$
$\mathcal{L}_0$	$\mathcal{L}_0$	$\mathcal{L}_a$	$\mathcal{L}_{pt}$	$\mathcal{L}_{apt}$
$\mathcal{L}_a$	$\mathcal{L}_a$	$\mathcal{L}_0$	$\mathcal{L}_{apt}$	$\mathcal{L}_{pt}$
$\mathcal{L}_{pt}$	$\mathcal{L}_{pt}$	$\mathcal{L}_{apt}$	$\mathcal{L}_0$	$\mathcal{L}_a$
$\mathcal{L}_{apt}$	$\mathcal{L}_{apt}$	$\mathcal{L}_{pt}$	$\mathcal{L}_a$	$\mathcal{L}_0$

group.[10,11] We denote this group by G_L. The generator of this group is the set $\{\mathcal{L}_0, \mathcal{L}_a, \mathcal{L}_p, \mathcal{L}_t\}$. The trajectories that can be generated by the group elements (operators) from a given nominal extremal trajectory are symmetric to the nominal one, in the general algebraic sense as defined in Sec. III. Indeed, G_L is homomorphic to the group of symmetry transformations with the operation composition (denoted G_M).

For our purpose here, rather than the multiplication table of the whole group G_L, we present a subgroup of particular significance (Table 1). All of the operators from this subgroup generate symmetric trajectories that perform the same +90-deg roll maneuver as the nominal extremal trajectory. Thus, by applying the subgroup operators to one particular extremal solution, we can obtain three more extremals with no additional numerical labor.

As an example, three families of extremal trajectories are presented in Figs. 1–8. The plots in Figs. 1a and 1b correspond to a nominal extremal trajectory M_0. The trajectories in Figs. 2a and 2b, 3a and 3b, and 4a and 4b correspond to the M_a, M_{pt}, and M_{apt} trajectories and are obtained by the transformations $\mathcal{L}_a$, $\mathcal{L}_{pt}$, and $\mathcal{L}_{apt}$, respectively. In a similar manner, the extremal trajectory in Figs. 6a and 6b is derived from the extremal trajectory shown in Figs. 5a and 5b by applying the transformation $\mathcal{L}_a$. In this case, however, $M_{pt} = M_0$ and $M_{apt} = M_a$, due to the inherent time symmetry of the extremal motions. The same is the case with the extremals shown in Figs. 7a and 7b and 8a and 8b.

VI. Conclusions

In the absence of an atmosphere, vehicle attitude dynamics exhibit certain peculiarities that can be employed for analytical derivation of new extremal solutions from given known ones. The analytical relations derived are based on the concepts of generalized algebraic symmetries and composition of symmetry transformations. Lurking in the shadow of the analytical derivations presented is the process of analysis of the underlying dynamical system.

Whereas the presentation was restricted to regular minimum-time-optimal extremals, one can easily extend the ideas to both regular and/or singular extremal trajectories with a broad class of other performance indices (the ones that are invariant on the sign change and/or time reversal in the states and controls.

The ideas can be also extended to reorientation maneuvers with respect to each of the body axes, in which case the Euler parameters (quaternions) are a better choice for attitude specification, due to the inherent symmetry they exhibit. Furthermore, some necessary conditions that adjoint variables for each particular trajectory need to satisfy for certain reorientation maneuvers (a subject not touched on in the present paper) can be shown. These additional necessary conditions can be utilized for reduction of the number of unknowns involved in the numerical search for extremal trajectories.

It was not the intention for this work to provide a high level of generality. Rather, we describe a convenient procedure used in one particular step of one particular problem (combat aircraft fuselage reorientation maneuvers). The benefit was immeasurable, however, and the message of this paper is that analysis of the structure of a dynamical system can provide a wealth of useful information to help complement the numerical effort.

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